THE CALCULATION OF LIFE OFFICE PREMIUMS

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It may fairly be claimed that the most vital requirement in life insurance is that the premiums charged shall be adequate, and, accordingly, an understanding of the principles which govern their calculation is essential. Since life insurance as represented by fraternal and assessment companies will be dealt with elsewhere in this volume, this paper proposes to deal only with the premiums of old line companies.

Scientific Basis.—The basis of life insurance is the doctrine of average and the theory of probability, and premiums are calculated from what are called life tables, formed by combining tables of mortality and tables of compound interest.

Compound Interest.—In the calculation of premiums it is almost invariably assumed that interest is compounded annually, and in the following investigations this assumption will be made.

Let P denote the principal sum and i the rate of interest. The amount of P at the end of one year is $P \times (1 + i)$; at the end of two years, $P \times (1 + i) \times (1 + i) = P\ (1 + i)^2$, and if S denote the amount to which P has accumulated at the end of n years, we have

$$S = P\ (1 + i)^n \qquad (1)$$

Let v denote the present value of 1 due at the end of a year, then since 1 amounts at the end of a year to $(1 + i)$, therefore $\frac{1}{1 + i}$ will amount at the end of a year to 1 and

$$v = \frac{1}{1 + i} \qquad (2)$$

Similarly, if v^n represents the present value of 1 due at the end of n years

$$v^n = \frac{1}{(1 + i)^n} \tag{3}$$

Let d be the discount on 1 for a year, then d is equal to the difference between the sum due and its present value, or

$$d = 1 - v = 1 - \frac{1}{1 + i} = \frac{i}{1 + i} = vi \tag{4}$$

Also, since $d = 1 - v$ therefore $v = 1 - d$. (5)

Mortality Tables.—A mortality table has been defined as the instrument by means of which are measured the probabilities of life and the probabilities of death. It has been shown by various investigations that the chance of death at any age of a person in average health is a very definite quantity. The exact date of death of the individual cannot be foretold, but if we have a large number of persons of the same age it is possible to foretell with great accuracy that a certain number will probably die in the first year, a certain number in the second year, and so on until all are dead. It must be kept very clearly in mind that the science of life insurance is based on the law of averages, and when we hear such expression as that the expectation of life of a person aged forty is 28.18 years, it does not mean that an individual aged forty may expect to live exactly 28.18 years, but only that if a large number of persons were traced from age forty to the end of life, their average lifetime from forty until death would approximate to 28.18 years.

A table of mortality contains the following columns: first, a column showing the number of persons living at each age—l_x; second, a column showing the number who die at each age—d_x.

Let us take the American table of mortality for illustration. At age ten, the commencing age of the table, an arbitrary number, 100,000, was taken, and the first column, l_x, shows opposite each age the number who survive at each age out of this original 100,000.

The second column, d_x, gives the number who die at each age, and can be got by differencing the numbers in the living column; for example, at age forty the number living is 78,106, and at age forty-one the survivors number 77,341; the difference between these

numbers, 765, is the number who die in the year of age forty; thus we have,

$$l_{40} - l_{41} = d_{40}$$

The number of survivors gradually diminish to three at age ninety-five, and as three are assumed to die in the following year, there are no survivors at age ninety-six, which is the limiting age of the table.

A table of mortality is constructed by tabulating for as large a number of persons as possible, the ages at the time the observations commence, the period of time during which the life is under observation, and the number of deaths occurring at each age. Then the number of lives who attain each age is tabulated opposite each age, and the number of deaths is put opposite the age at which the death occurs. Next the number of deaths opposite each age is divided by the number living at that age and we obtain the probability of death at individual ages, denoted by q_x. This value is usually somewhat irregular and is smoothed by a graduation formula. Then the arbitrary number assumed to be living at the initial age of the table, called the radix, is multiplied by the probability of dying at that age, giving the number to be put in the d_x column, this number of deaths is subtracted from the 100,000, and the number left surviving, 99,251, is multiplied by the probability of dying in a year at age eleven, and so on to the limit of the table. The following extract from the American Table of Mortality will show how the number of survivors diminish and how the death-rate runs:

Age.	Number Living.	Deaths.	Probability of Death in a Year.
10	100,000	749	.007490
20	92,637	723	.007805
30	85,441	720	.008427
40	78,106	765	.009794
50	69,804	962	.013781
70	38,569	2,391	.061993
90	847	385	.454545
91	462	246	.532468
92	216	137	.634259
93	79	58	.734177
94	21	18	.857143
95	3	3	1.000000
96	0	0	

Probabilities of Life and Death.—By means of the theory of probabilities we measure the chance of an event happening, and the most convenient way of representing this chance mathematically is by means of a fraction.

By an examination of the American table, we see that of the 100,000 lives who start at age ten, 69,804 survive to age fifty, thus the probability that a person aged ten will attain age fifty is $\frac{69,804}{100,000}$

The probability that a person aged x will live n years is denoted by ${}_np_x$, and that a person aged x will die within n years is denoted by ${}_nq_x$.

$$ {}_np_x = \frac{l_{x+n}}{l_x} \tag{6} $$

$$ {}_nq_x = \frac{l_x - l_{x+n}}{l_x} \tag{7} $$

Equation (7) may be explained as follows: At age x the number living is l_x, at age $x + n$ the number of living is l_{x+n}, the difference $(l_x - l_{x+n})$ gives the number who die between ages x and $x + n$, and dividing this by l_x, the original number living, gives the probability that a person age x will die in the next n years.

Life Annuity.—To find the value of 1 to be received by a person now aged x provided he live to the end of a year, we have to multiply the present value of 1 due at the end of a year by the probability that (x), which we will understand to mean a person aged x, will be living at the end of a year, and we obtain from (6) the result $v \cdot \frac{l_{x+1}}{l_x}$, if the amount is to be received at the end of n years, only in event of survival, the value, called a pure endowment, is

$$ {}_nE_x = v^n \cdot \frac{l_{x+n}}{l_x} \tag{8} $$

A life annuity is a periodical payment of money to be made to a person provided such person is alive at the time the payment falls due. It is, in fact, a series of pure endowments, payable at the end of successive years, the consideration being a cash payment down.

Giving to n in (8) all values from 1 upwards and again assuming the present age to be x, we have the value of a life annuity,

$$a_x = \frac{vl_{x+1}}{l_x} + \frac{v^2 l_{x+2}}{l_x} + \frac{v^3 l_{x+3}}{l_x} + \cdots\cdots \tag{9}$$

From this we see that the value of a life annuity may be calculated by multiplying the factors contained in an interest table and a mortality table, but the process would be very laborious. By means of a simple device much labor is saved. Let us multiply both numerator and denominator of the right-hand side of (9) by v^x, then we have

$$a_x = \frac{v^{x+1}\, l_{x+1} + v^{x+2}\, l_{x+2} + v^{x+3}\, l_{x+3} + \cdots\cdots}{v^x\, l_x} \tag{10}$$

Denote $v^x\, l_x$ by D_x and we obtain

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \cdots\cdots}{D_x} \tag{11}$$

Again, let the summation of $D_x + D_{x+1} + D_{x+2} + \cdots\cdots$ to the limit of the table be denoted by $\boldsymbol{N}_x$ and we finally have

$$a_x = \frac{\boldsymbol{N}_{x+1}}{D_x} \tag{12}$$

The form of the $\boldsymbol{N}$ column where $\boldsymbol{N}_x$ represents the summation of $D_x + D_{x+1} + D_{x+2} + \cdots\cdot$ must be carefully distinguished from the form $N_x = D_{x+1} + D_{x+2} + \cdots\cdot$ which was adopted by the International Congress of Actuaries, but has not been received with much favor in the United States, where the form $\boldsymbol{N}_x = D_x + D_{x+1} + \ldots.$ has been found more convenient.

The values D and $\boldsymbol{N}$ are tabulated in columns, and with other values to be mentioned later, form what are called commutation columns.

These values have no special meaning in themselves, but their tabulation has enormously reduced the labor of life insurance calculations.

If the first payment of the annuity is to be made at once we have

$$\mathbf{a}_x = 1 + a_x = \frac{N_x}{D_x} \tag{13}$$

We can now express the value of a pure endowment to (x) at the end of n years as follows:

$$_nE_x = \frac{D_{x+n}}{D_x} \tag{14}$$

Temporary Annuity.—If the annuity is to consist of n payments only, and the first payment is to be immediate, we obtain

$$\mathbf{a}_{x\overline{n}|} = 1 + \frac{vl_{x+1}}{l_x} + \frac{v^2\, l_{x+2}}{l_x} + \cdots\cdots + \frac{v^{n-1}\, l_{x+n-1}}{l_x}$$

$$= \frac{v^x\, l_x + v^{x+1}\, l_{x+1} + v^{x+2}\, l_{x+2} + \cdots + v^{x+n-1}\, l_{x+n-1}}{v^x\, l_x}$$

$$= \frac{D_x + D_{x+1} + \cdots\cdots + D_{x+n-1}}{D_x}$$

$$= \frac{N_x - N_{x+n}}{D_x} \tag{15}$$

This is called a temporary life annuity and is used in calculating annual premiums where the period of premium payments is limited to a term of years.

Ordinary Whole Life Insurance.—To find the value of 1 payable at the end of the year of death of (x), the first year the value of the insurance is $v \cdot \frac{d_x}{l_x}$ the second year $v^2 \cdot \frac{d_{x+1}}{l_x}$, and so on.

Denote the single premium for an ordinary life insurance of 1 by A_x, and we have

$$A_x = \frac{vd_x + v^2\, d_{x+1} + v^3\, d_{x+2} + \cdots}{l_x} \tag{16}$$

Again multiplying above and below by v^x and denoting $v^{x+1} \cdot d_x$

by C_x and the summation $C_x + C_{x+1} + C_{x+2} + \cdots\cdots$ by M_x, we obtain

$$A_x = \frac{v^{x+1}d_x + v^{x+2}d_{x+1} + v^{x+3}d_{x+2} + \cdots\cdots}{v^x l_x} \tag{17}$$

$$= \frac{C_x + C_{x+1} + C_{x+2} + \cdots\cdot}{D_x} \tag{18}$$

$$= \frac{M_x}{D_x} \tag{19}$$

The annual premium for an ordinary life insurance is payable for the whole of life and to find the annual equivalent for the single premium we have the following equation, in which we equate the benefit to payment,

$$P_x(1 + a_x) = A_x$$

$$\text{therefore} \quad P_x = \frac{A_x}{1 + a_x} = \frac{\dfrac{M_x}{D_x}}{\dfrac{N_x}{D_x}} = \frac{M_x}{N_x} \tag{20}$$

Temporary Insurance.—In this form, the sum insured is payable only if death occur within a certain number of years. The value of a temporary life annuity was shown by (15) to be $\frac{N_x - N_{x+n}}{D_x}$.

Similarly, it may be shown that the value of a temporary insurance,

$$A^{1}_{x\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} \tag{21}$$

and the annual premium for a temporary insurance,

$$P^{1}_{x\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}} \tag{22}$$

Limited Payment Life.—If the sum insured is to be payable at death, but the premium is to be payable for a certain number of

years only, we have what is known as a limited payment policy, the annual premium,

$$ {}_nP_x = \frac{M_x}{N_x - N_{x+n}} \tag{23} $$

Endowment.—This is a combination of a temporary insurance and a pure endowment, the amount insured being payable either in event of death within n years or in event of survival at the end of n years. The formula is obtained by combining (14) and (22), and denoting the annual premium for an endowment by $P_{\overline{xn}|}$ we get

$$ P_{\overline{xn}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}} \tag{24} $$

It will be noticed that the annual premium is obtained by dividing the single premium by a temporary annuity for the term for which the premium is payable, and that the first payment of the annuity is taken as payable at once. This applies to all premiums and is due to the fact that life insurance premiums are always payable at the beginning of the policy year.

For most of the standard tables the net premiums have been tabulated at each age for the ordinary forms of policies and are available without the necessity of making any calculation. We have shown how they may be calculated for these simple forms. By the construction of other commutation columns, values may be obtained for other and more complex benefits, but it is beyond the scope of the present paper to go more deeply into the mathematical part of the subject.

Reserve or Policy Value.—The reserve on a policy is the sum which the office must have in hand to provide for the future liability under a contract which is not covered by the value of the premiums still to be received, and the only source from which it can be derived is the accumulation of the portions of the net premiums already received and not used in the past for the cost of insurance. The risk of death increases with advancing age, and taking the case of a whole life policy with level annual premiums the excess payments in the early years must be husbanded to meet the time when the cost of carrying the insurance exceeds the premium payable. In an endowment policy an additional amount has to be set aside towards the payment of the endowments which will fall due by

reason of the survival of the life insured to the end of the endowment term. The lower the interest assumption, the higher will be the reserve. For instance: a reserve based on 3 per cent. is larger than one based on 3½ per cent., for the reason that under the first assumption the company only needs to earn 3 per cent. on its investments in order to fulfill its contract, whereas under the latter a minimum return of 3½ per cent. interest is necessary. It is apparent that a larger sum must be set aside each year out of the premium where the amount is to be invested at the lower rate. The use of a stringent interest rate in computing reserves has resulted in the companies charging higher rates than those in use some years ago, but this increase in rate has been counter-balanced to some extent by the much more liberal guarantees now contained in a policy as regards cash and loan values, paid-up insurance, and the other non-forfeiture provisions in event of discontinuance of the premium payments.

Standard Mortality Tables.—In this country two mortality tables are in common use, the Actuaries', or combined table of mortality, and the American experience table of mortality.

The Actuaries' table was published in 1843, and was constructed from statistics of insured lives furnished by seventeen (17) English offices. It is now admitted that the mortality shown by this table is much heavier than that experienced by well-managed life offices, and it is now being rapidly superseded by the American table, which has come to be looked upon as the standard table of the United States.

The American experience table of mortality was formed by Sheppard Homans, and was first published about the year 1870. The statistics deduced from the experience of the "Mutual Life" of New York were the bases of the table, but considerable adjustments were made, especially at the older ages. The table gives a very good indication of the mortality likely to be experienced among insured lives after the initial effects of the medical selection have worn off, and the table is now very generally used. Most of the state insurance departments have adopted the American table as their standard for the calculation of the reserve liability of the companies.

In the calculation of rates for life annuities it is desirable to use a table which distinguishes between male and female lives.

Below the age of fifty, the mortality amongst females has been found to be heavier than of males, but at ages above fifty the superior vitality of the female is very pronounced, and nearly all life annuities are issued at ages over fifty. Recent returns of the life companies would seem to indicate that annuity business is run on a very narrow margin and an authoritative table, showing the mortality amongst American annuitants distinguishing between male and female lives, would be welcomed by all actuaries. The growing accumulation of wealth in this country and the difficulty of obtaining high rates of interest in return for investment of capital make it likely that the purchase of life annuities will largely increase in the near future. At present, probably the best table available for life annuity calculation is that recently issued by the British Institute of Actuaries, giving the mortality amongst annuitants in the English and Scotch companies.

Interest Basis.—The usual rate of interest adopted for the calculation of premiums at the present time is either 3 per cent. or 3½ per cent. Prior to 1901, the state valuation requirements were mostly based on a 4 per cent., but since that time several of the states required a 3½ per cent standard on all new business written, and many of the strongest companies have gone a step further and base their rates on 3 per cent. The following table of net premiums at age thirty-five, by the American experience table of mortality illustrates the difference in the net annual premiums according as the interest basis is taken at 3 per cent. or 3½ per cent.

	3% Basis.	3½% Basis.
Ordinary Life	$21.08	$19.91
10 Payment Life	49.73	44.78
20 Payment Life	29.85	27.40
10 Year Endowment	89.30	87.02
20 Year Endowment	41.97	40.12

In view of the fall in the rate of interest obtainable in recent years on first-class securities it would appear that the rate of interest to be assumed for the future should not exceed 3½ per cent., and that 3 per cent. is a very suitable rate for use in the calculation of life insurance premiums for policies which are to participate in the surplus earnings of the company. If the company earns a higher rate of interest than that assumed in determining its reserve liability,

which is usually calculated at the same rate of interest as that used in computing the premiums, the excess interest receipts form a source of profit, and several companies have adopted a rate of 3 per cent. where the legal requirement is 3½ per cent., on the ground that it puts them in a stronger position and will enable them to earn a larger margin of surplus interest in years to come and thus help to give larger dividend returns to policy holders. For policies which do not participate in profits, a somewhat higher rate of interest may be used in computing premiums than that adopted in calculating the premiums on participating policies.

Loading.—We have now considered two of the important factors in rate making, the mortality table and the interest basis. The third and final function is the loading. Up to this point we have dealt with what are called net premiums, in which there is no provision for expenses or contingencies. In the procurement of new business and the care of the old, expense is necessarily incurred, and to provide for this an addition, called loading, is made to the net or mathematical premium. Returns to policy holders in the way of dividends have also come to be such a recognized part of the system of life insurance that the loading is usually made sufficient to insure a surplus. Where the company returns in the form of dividends the excess payments made over those required, it may be pointed out that the actual cost to the policy holder is sometimes reduced below the net premium, the insured member getting the benefit of the actual rate of interest earned, the actual rate of mortality experienced, and paying his just share of the expenses. Keeping this in view, we see that it is much better to have a rate too large than too small; because when the rate is larger than is absolutely necessary, an adjustment is made by the dividend distribution, whereas, if the rate is too small, disaster must follow. In theory, therefore, the results under a participating policy should be more favorable than under a non-participating policy, as in order to be safe the company must, under the latter form, have a small margin which will not be returned to the policy holder. The method of loading has varied from time to time, and there is still great diversity of opinion as to which is the best method. In early times, the usual method was simply to increase the net premium by a fixed percentage. This method had the disadvantage of making the loading very heavy at the old ages.

The commission payable is usually fixed at a certain percentage of the premium, and the method mentioned applies well as regards this item, but the other expenses, such as the cost of the medical examination, the writing of the policy, and the home office expenses, are just as heavy for a young age as for an old age. Another method would be to add a constant amount to the net premium irrespective of the age. This method has exactly the opposite effect to that produced by adding a percentage to the net premium, for it makes the loading relatively much heavier on the young than on the old ages.

A method which has come into great favor is to load the net premium for the particular contract with a certain percentage of itself and, in addition, add a certain percentage of the ordinary life net premium. For example, on a twenty year endowment, the gross or office premium might be made as follows, at the age of forty:

American 3% Net Premium for 20 Year Endowment........	$43.01
Add 12½%	5.38
Add 12½% of Net Ordinary Life Premium ($24.75).........	3.09
Gross Premium	$51.48

The following table shows the effect of a loading on the above basis as between different ages and different classes of insurance:

SPECIMEN RATES.

American 3% loaded 12½% plus 12½% of Net Ordinary Life Premium.

	Age.	Net Prem.	12½% Net Prem.	12½% Net Ord. Life Prem.	Total Loading.	Office Prem.	Percentage of loading to Net Prem.	Percentage of loading to Office Prem.
Ord. Life.								
	21	$14.72	$1.84	$1.84	$3.68	$18.40	25%	20%
	40	24.75	3.09	3.09	6.18	30.93	25%	20%
	60	58.27	7.28	7.28	14.56	72.83	25%	20%
10 Payt. Life.								
	21	39.52	4.94	1.84	6.78	46.30	17%	15%
	40	54.66	6.83	3.09	9.92	64.58	18%	15%
	60	87.22	10.90	7.28	18.18	105.40	21%	17%
20 Payt. Life.								
	21	23.48	2.94	1.84	4.78	28.26	20%	17%
	40	33.14	4.14	3.09	7.23	40.37	22%	18%
	60	61.62	7.70	7.28	14.98	76.60	24%	20%

10 Year Endt.	Age.	Net Prem.	12½% Net Prem.	12½% Net Ord. Life Prem.	Total Loading.	Office Prem.	Percentage of loading to Net Prem.	Percentage of loading to Office Prem.
	21	88.62	11.08	1.84	12.92	101.54	15%	13%
	40	89.86	11.23	3.09	14.32	104.18	16%	14%
	60	101.69	12.71	7.28	19.99	121.68	20%	16%
20 Year Endt.								
	21	40.81	5.10	1.84	6.94	47.75	17%	15%
	40	43.01	5.38	3.09	8.47	51.48	20%	16%
	60	63.29	7.91	7.28	15.19	78.48	24%	19%

An examination of the last column given above will show that although this method makes some adjustment as between different classes of policies, yet it does not adjust equitably the expenses between old and young entrants. The method has the advantage of being very elastic, as by varying the loading percentages of the contract net premium or of the ordinary life net premium, the gross premiums may be modified as desired. It has been argued by some actuaries that the rates for old entrants should be more heavily loaded, as it is believed that the medical selection is more efficacious in the case of young lives.

Constant and Percentage Loading.—A good method would be to add to the net premium a constant and then load this gross amount with a percentage. For illustration, let us take the constant as $3.00 per $1,000 insured, the percentage as 15 per cent., then using the American 3 per cent. table of mortality we get the following rates:

SPECIMEN RATES.

American 3% loaded with a constant of 3 per 1000, plus 15%.

Ord. Life.	Age.	Net Prem.	Constant.	15% of Net Prem. plus constant.	Total Loading.	Office Prem.	Percentage of loading to Net Prem.	Percentage of loading to Office Prem
	21	$14.72	$3.00	$2.66	$5.66	$20.38	39%	28%
	40	24.75	3.00	4.17	7.17	31.92	29%	23%
	60	58.27	3.00	9.19	12.19	70.46	21%	17%
10 Payt. Life.								
	21	39.52	3.00	5.93	8.93	48.45	23%	18%
	40	54.66	3.00	8.20	11.20	65.86	21%	17%
	60	87.22	3.00	13.08	16.08	103.30	18%	16%

	Age.	Net Prem.	Con-stant.	15% of Net Prem. plus constant.	Total Load-ing.	Office Prem.	Percentage of loading to Net Prem.	Percentage of loading to Office Prem.
20 Payt. Life.								
	21	23.48	3.00	3.97	6.97	30.45	30%	23%
	40	33.14	3.00	5.42	8.42	41.56	25%	20%
	60	61.62	3.00	9.69	12.69	74.31	21%	17%
10 Year Endt.								
	21	88.62	3.00	13.29	16.29	104.91	18%	16%
	40	89.86	3.00	13.48	16.48	106.34	18%	16%
	60	101.69	3.00	15.25	18.25	119.94	18%	15%
20 Year Endt.								
	21	40.81	3.00	6.57	9.57	50.38	23%	19%
	40	43.01	3.00	6.90	9.90	52.91	23%	19%
	60	63.29	3.00	9.94	12.94	76.23	20%	17%

This method of loading appears to fulfill our requirements somewhat better than the methods previously mentioned. While the amount of loading increases with an increase in age, yet the percentage decreases, and the equities between different classes of insurance are preserved. It will be understood the figures given are merely for the purpose of illustrating the different methods of loading, and that the percentage or constant employed must be fixed with regard to the actual expenses likely to be incurred.

In the practical calculation of office premiums, the rates of competing companies must be very carefully considered. Competition is very sharp, and the rates of the large companies form a standard from which it is not judicious to depart very far. The distribution of dividends, which is now almost universally made having regard to the amount contributed by each individual, removes apparent inequalities. Amongst other considerations which enter into the method of calculating office premiums, perhaps the most important is the amount of surrender value to be allowed in event of discontinuance, but this phase of the subject is too large to be included in the paper now presented.